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# Momentum, Angular Momentum, and Equations of Motion for Test Bodies in Space-Time with Torsion

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The definitions and transformation properties of momentum and angular momentum of test bodies possessing both macroscopic rotation and net spin are discussed. The equations of motion for momentum and angular momentum of test bodies are derived and written in a covariant form when the energy-momentum tensor is symmetric.

## 1. INTRODUCTION

In a previous paper (Chen, 1990) (hereafter referred to as I), the author derived the following conservation laws (or identities) for matter fields from the most general functional form of Lagrangian density $\mathscr{L}_M$ in space-time with torsion:

$$\mathfrak{T}_\nu{}^\mu{}_{,\mu} - \Gamma^\sigma{}_{\mu\nu}\mathfrak{T}_\sigma{}^\mu - \tfrac{1}{2}\mathfrak{C}_{ij}{}^\mu R^{ij}{}_{\mu\nu} = 0 \tag{1}$$

$$\frac{\partial}{\partial x^\mu}\mathfrak{C}_{ij}{}^\mu = 2\mathfrak{T}_{[ij]} + \Gamma^k{}_{i\mu}\mathfrak{C}_{kj}{}^\mu + \Gamma^k{}_{j\mu}\mathfrak{C}_{ik}{}^\mu \tag{2}$$

where

$$\mathfrak{T}_\nu{}^\mu = h^i{}_\nu \mathfrak{T}_i{}^\mu, \qquad \mathfrak{T}_{ij} = \eta_{kj} h^k{}_\mu \mathfrak{T}_i{}^\mu$$

$$\mathfrak{T}_i{}^\mu = \frac{\partial \mathscr{L}_M}{\partial h^i{}_\mu} - \frac{\partial}{\partial x^\lambda}\left(\frac{\partial \mathscr{L}_M}{\partial h^i{}_{\mu,\lambda}}\right)$$

$$\mathfrak{C}_{ij}{}^\mu = -2\left[\frac{\partial \mathscr{L}_M}{\partial \Gamma^{ij}{}_\mu} - \frac{\partial}{\partial x^\lambda}\left(\frac{\partial \mathscr{L}_M}{\partial \Gamma^{ij}{}_{\mu,\lambda}}\right)\right]$$

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All of $\mathfrak{T}_i{}^\mu$, $\mathfrak{T}_\nu{}^\mu$, and $\mathfrak{T}_{ij}$ are energy-momentum tensor densities of matter fields, but in different indices, and $\mathfrak{C}_{ij}{}^\mu$ is called the generalized spin density in I; $i, j, k$, and other Latin letters are frame indices; $\mu$, $\nu$, $\lambda$, and other Greek letters are coordinate indices. $\mathfrak{T}^{\nu\mu}$ $(=g^{\nu\sigma}\mathfrak{T}_\sigma{}^\mu)$ or $\mathfrak{T}_{ij}$ may be asymmetric in the general case.

The aim of this paper is to derive the equations of motion for test bodies from equations (1) and (2). I shall treat the complex case in which the test bodies (e.g., neutron stars) may possess both macroscopic rotation and net spin.

Papapetrou (1951) derived a set of equations of motion for test bodies; his equations are covariant with respect to the general coordinate transformations. However, Papapetrou's equations can only describe the behavior of test bodies possessing macroscopic rotation but without net spin, and those equations are suited only to the space-time of general relativity in which torsion does not exist. Yasskin and Stoeger (1980) derived another set of equations of motion for test bodies possessing both macroscopic rotation and net spin in space-time with torsion, but their equations are not written in a covariant form, i.e., they are suited only to special coordinates. The equations of motion derived in this paper are more general and simpler than Papapetrou's equations and Yaskin and Stoeger's equations.

## 2. SOME DEFINITIONS AND RELATIONS

The test body can be looked upon as a macroscopic matter field whose dimensions in 3-space are very small compared with a certain characteristic length; thus, the test body will describe a narrow tube in the space-time. According to the method developed by Papapetrou (1951), a line $L$ which may represent the motion of the test body is chosen inside this tube. The coordinates of this line are denoted by $X^\alpha$, they are functions of the proper time $S$ along the line $L$, where $dS = g_{\mu\nu}\, dX^\mu\, dX^\nu$. The tensor density fields $\mathscr{D}^{\mu\nu\cdots}_{\alpha\beta\cdots}(x)$ which describe certain properties of the test body are different from zero only inside the world tube.

Define the quantities $Q$ from $\mathscr{D}$ by the integrals

$$^{\lambda_1\lambda_2\cdots\lambda_n}Q^{\mu\nu\cdots}_{\alpha\beta\cdots}(X) = U^0 \int \delta x^{\lambda_1}\delta x^{\lambda_2}\cdots\delta^{\lambda_n}\mathscr{D}^{\mu\nu\cdots}_{\alpha\beta\cdots}(x)\, d^3x \tag{3}$$

where

$$\delta x^\lambda = x^\lambda - X^\lambda, \qquad U^0 = \frac{dX^0}{dS}$$

The integration is carried out over the three-dimensional volume at a constant $t = X^0$ (the speed of light is taken as unity). The quantities

$$^{\lambda_1\lambda_2\cdots\lambda_n}Q^{\mu\nu\cdots}_{\alpha\beta\cdots}(X)$$

are called the moments of order $n$ for the field $\mathscr{D}^{\mu\nu\cdots}_{\alpha\beta\cdots}(x)$.

Thus, the moments of order 0 and 1 for the energy-momentum tensor density field $\mathfrak{T}^{\nu\mu}(x)$ are defined, respectively, by

$$M^{\nu\mu}(X) = U^0 \int \mathfrak{T}^{\nu\mu}(x)\, d^3x \tag{4}$$

and

$$^{\lambda}M^{\nu\mu}(X) = U^0 \int \delta x^{\lambda}\, \mathfrak{T}^{\nu\mu}(x)\, d^3x \tag{5}$$

Note that $^{0}M^{\nu\mu}(X) = 0$ since the integral refers to the hyperplane $t = X^0$ and $\delta x^0 = 0$. Moreover, the moment of order 0 for the generalized spin density field $\mathfrak{C}^{\lambda\nu\mu}(x)$ is defined by

$$N^{\lambda\nu\mu}(X) = U^0 \int \mathfrak{C}^{\lambda\nu\mu}(x)\, d^3x \tag{6}$$

where

$$\mathfrak{C}^{\lambda\nu\mu}(x) = g^{\lambda\sigma} g^{\nu\tau} h^{i}_{\sigma} h^{j}_{\tau} \mathfrak{C}_{ij}{}^{\mu}(x)$$

Usually the macroscopic rotation angular momentum and generalized spin angular momentum of a test body are defined by

$$L^{\lambda\nu}(X) = \frac{^{\lambda}M^{\nu 0}(X) - {}^{\nu}M^{\lambda 0}(X)}{U^0} = \int [\delta x^{\lambda} \mathfrak{T}^{\nu 0}(x) - \delta x^{\nu} \mathfrak{T}^{\lambda 0}(x)]\, d^3x \tag{7}$$

and

$$S^{\lambda\nu}(X) = \frac{N^{\lambda\nu 0}(X)}{U^0} = \int \mathfrak{C}^{\lambda\nu 0}(x)\, d^3x \tag{8}$$

respectively. In I, $S^{\lambda\nu}$ is denoted by $C^{\lambda\nu}$. Since the three space dimensions of the test body are very small and the rotation angular momentum is greater than the spin angular momentum in general for rotational bodies (e.g., neutron stars), it is supposed that all moments of order higher than 1 for the field $\mathfrak{T}^{\nu\mu}(x)$ and all moments of order higher than 0 for the field $\mathfrak{C}^{\lambda\nu\mu}(x)$ are vanishingly small and can be neglected.

By raising, lowering, or altering the indices of the tensor density, we can rewrite equations (1) and (2) as

$$\mathfrak{T}^{\nu\mu}{}_{,\mu} + (\Gamma^{\nu}{}_{\sigma\mu} - 2T_{\sigma\mu}{}^{\nu})\mathfrak{T}^{\sigma\mu} - \tfrac{1}{2}\mathfrak{C}^{\alpha\beta\mu}R_{\alpha\beta\mu}{}^{\nu} = 0 \tag{1'}$$

and

$$\mathfrak{C}^{\alpha\beta\mu}{}_{,\mu} = 2\mathfrak{T}^{[\alpha\beta]} - \Gamma^{\alpha}{}_{\sigma\mu}\mathfrak{C}^{\sigma\beta\mu} - \Gamma^{\beta}{}_{\sigma\mu}\mathfrak{C}^{\alpha\sigma\mu} \tag{2'}$$

where $T_{\sigma\mu}{}^{\nu} = g_{\alpha\sigma}g^{\nu\beta}T^{\alpha}{}_{\mu\beta}$ and $T^{\alpha}{}_{\mu B}$ is the torsion tensor.

Utilizing the relation

$$\Gamma^{\nu}{}_{\sigma\mu} = \begin{Bmatrix} \nu \\ \sigma\mu \end{Bmatrix} + T^{\nu}{}_{\sigma\mu} - T_{\sigma}{}^{\nu}{}_{\mu} - T_{\mu}{}^{\nu}{}_{\sigma}$$

we find that (1′) becomes

$$\mathfrak{T}^{\nu\mu}{}_{,\mu} + \left(\begin{Bmatrix} \nu \\ \sigma\mu \end{Bmatrix} - A^{\nu}{}_{\sigma\mu}\right)\mathfrak{T}^{\sigma\mu} - \tfrac{1}{2}\mathfrak{C}^{\alpha\beta\mu}R_{\alpha\beta\mu}{}^{\nu} = 0 \tag{1''}$$

where

$$A^{\nu}{}_{\sigma\mu} = T^{\nu}{}_{\sigma\mu} + T_{\sigma}{}^{\nu}{}_{\mu} - T_{\mu}{}^{\nu}{}_{\sigma} = -A^{\nu}{}_{\mu\sigma} \tag{9}$$

Using methods similar to that of Papapetrou (1951), it is not difficult to derive the following relations from (1″) and (2′):

$$\frac{d}{dS}\left(\frac{M^{\nu 0}}{U^0}\right) + \left(\begin{Bmatrix} \nu \\ \sigma\mu \end{Bmatrix} + A^{\nu}{}_{\sigma\mu}\right)M^{\sigma\mu} + \frac{\partial}{\partial X^{\lambda}}\left(\begin{Bmatrix} \nu \\ \sigma\mu \end{Bmatrix} + A^{\nu}{}_{\sigma\mu}\right){}^{\lambda}M^{\sigma\mu}$$
$$-\tfrac{1}{2}R_{\alpha\beta\mu}{}^{\nu}(U^{\mu}S^{\alpha\beta} + {}^{\mu}M^{[\alpha\beta]}) = 0 \tag{10}$$

$$M^{\nu\mu} = U^{\mu}\frac{M^{\nu 0}}{U^0} + \frac{d}{dS}\left(\frac{{}^{\mu}M^{\nu 0}}{U^0}\right) + \left(\begin{Bmatrix} \nu \\ \sigma\tau \end{Bmatrix} + A^{\nu}{}_{\sigma\tau}\right){}^{\mu}M^{\sigma\tau} \tag{11}$$

$${}^{\alpha}M^{\nu\beta} + {}^{\beta}M^{\nu\alpha} = \frac{U^{\alpha}}{U^0}({}^{\beta}M^{\nu 0}) + \frac{U^{\beta}}{U^0}({}^{\alpha}M^{\nu 0}) \tag{12}$$

$$\frac{d}{dS}S^{\alpha\beta} = 2M^{[\alpha\beta]} - \Gamma^{\alpha}{}_{\sigma\mu}N^{\sigma\beta\mu} - \Gamma^{\beta}{}_{\sigma\mu}N^{\alpha\sigma\mu} \tag{13}$$

$$N^{\alpha\beta\mu} = U^{\mu}S^{\alpha\beta} + 2^{\mu}M^{[\alpha\beta]} \tag{14}$$

where

$$U^{\mu} = \frac{dX^{\mu}}{dS}$$

I shall utilize these relations to find the equations of motion for test bodies. But before doing so, it is worth discussing the transformation properties of $M^{\nu\mu}(X)$, ${}^{\lambda}M^{\nu\mu}(X)$, and $N^{\lambda\nu\mu}(X)$.

## 3. THE TRANSFORMATION PROPERTIES OF $M^{\nu\mu}$, ${}^{\lambda}M^{\nu\mu}$, $N^{\lambda\nu\mu}$, $L^{\lambda\nu}$, AND $S^{\lambda\nu}$

In Appendix A I prove that under the infinitesimal coordinate transformations $x^{\mu} \rightarrow x'^{\mu} = x^{\mu} + \xi^{\mu}(x)$ the quantities

$$Q^{\mu_1\mu_2\cdots}_{\alpha_1\alpha_2\cdots}(X) \qquad \text{and} \qquad {}^{\lambda}Q^{\mu_1\mu_2\cdots}_{\alpha_1\alpha_2\cdots}(X)$$

transform according to

$$\begin{aligned} Q^{\mu_1\mu_2\cdots}_{\alpha_1\alpha_2\cdots}(X) = & \frac{\partial X^{\mu_1}}{\partial X'^{\nu_1}} \frac{\partial X^{\mu_2}}{\partial X'^{\nu_2}} \cdots \frac{\partial X'^{\beta_1}}{\partial X^{\alpha_1}} \frac{\partial X'^{\beta_2}}{\partial X^{\alpha_2}} \cdots Q'^{\nu_1\nu_2\cdots}_{\beta_1\beta_2\cdots}(X') \\ & + \frac{\partial}{\partial X'^{\lambda}} \left( \frac{\partial X^{\mu_1}}{\partial X'^{\nu_1}} \frac{\partial X^{\mu_1}}{\partial X'^{\nu_2}} \cdots \frac{\partial X'^{\beta_1}}{\partial X^{\alpha_1}} \frac{\partial X'^{\beta_2}}{\partial X^{\alpha_2}} \cdots \right) {}^{\lambda}Q'^{\nu_1\nu_2\cdots}_{\beta_1\beta_2\cdots}(X') \\ & - \frac{d}{dS'} \left( \frac{1}{U^0} \frac{\partial X^0}{\partial X'^{\lambda}} \frac{\partial X^{\mu_1}}{\partial X'^{\nu_1}} \frac{\partial X^{\mu_2}}{\partial X'^{\nu_2}} \cdots \frac{\partial X'^{\beta_1}}{\partial X^{\alpha_1}} \right. \\ & \left. \times \frac{\partial X'^{\beta_2}}{\partial X^{\alpha_2}} \cdots {}^{\lambda}Q'^{\nu_1\nu_2\cdots}_{\beta_1\beta_2\cdots}(X') \right) \end{aligned} \tag{15}$$

and

$$\begin{aligned} {}^{\lambda}Q^{\mu_1\mu_2\cdots}_{\alpha_1\alpha_2\cdots}(X) = & \frac{\partial X^{\mu_1}}{\partial X'^{\nu_1}} \frac{\partial X^{\mu_2}}{\partial X'^{\nu_2}} \cdots \frac{\partial X'^{\beta_1}}{\partial X'^{\alpha_1}} \frac{\partial X'^{\beta_2}}{\partial X'^{\alpha_2}} \cdots \\ & \times \left( \frac{\partial X^{\lambda}}{\partial X'^{\rho}} - \frac{U^{\lambda}}{U^0} \frac{\partial X^0}{\partial X'^{\rho}} \right) {}^{\rho}Q'^{\nu_1\nu_2\cdots}_{\beta_1\beta_2\cdots}(X') \end{aligned} \tag{16}$$

respectively, it is assumed here that all moments of order higher than 1 are equal to zero.

The transformation formulas of $M^{\nu\mu}$, ${}^{\lambda}M^{\nu\mu}$, and $N^{\lambda\nu\mu}$ can be obtained from (15) and (16) immediately:

$$\begin{aligned} M^{\nu\mu}(X) = & \frac{\partial X^{\nu}}{\partial X'^{\alpha}} \frac{\partial X^{\mu}}{\partial X'^{\beta}} M'^{\alpha\beta}(X') + \frac{\partial}{\partial X'^{\lambda}} \left( \frac{\partial X^{\nu}}{\partial X'^{\alpha}} \frac{\partial X^{\mu}}{\partial X'^{\beta}} \right) {}^{\lambda}M'^{\alpha\beta}(X') \\ & - \frac{d}{dS} \left( \frac{1}{U^0} \frac{\partial X^0}{\partial X'^{\lambda}} \frac{\partial X^{\nu}}{\partial X'^{\alpha}} \frac{\partial X^{\mu}}{\partial X'^{\beta}} {}^{\lambda}M'^{\alpha\beta}(X') \right) \end{aligned} \tag{17}$$

$$ {}^{\lambda}M^{\nu\mu}(X)=\frac{\partial X^{\nu}}{\partial X'^{\alpha}}\frac{\partial X^{\mu}}{\partial X'^{\beta}}\left(\frac{\partial X^{\lambda}}{\partial X'^{\rho}}-\frac{U^{\lambda}}{U^{0}}\frac{\partial X^{0}}{\partial X'^{\rho}}\right){}^{\rho}M'^{\alpha\beta}(X') \tag{18} $$

$$ \begin{aligned} N^{\lambda\nu\mu}(X)=&\frac{\partial X^{\lambda}}{\partial X'^{\sigma}}\frac{\partial X^{\nu}}{\partial X'^{\alpha}}\frac{\partial X^{\mu}}{\partial X'^{\beta}}N'^{\sigma\alpha\beta}(X')+\frac{\partial}{\partial X'^{\rho}}\left(\frac{\partial X^{\lambda}}{\partial X'^{\sigma}}\frac{\partial X^{\mu}}{\partial X'^{\alpha}}\frac{\partial X^{\mu}}{\partial X'^{\beta}}\right){}^{\rho}N'^{\sigma\alpha\beta}(X')\\ &-\frac{d}{dS'}\left(\frac{1}{U^{0}}\frac{\partial X^{0}}{\partial X'^{\rho}}\frac{\partial X^{\lambda}}{\partial X'^{\sigma}}\frac{\partial X^{\nu}}{\partial X'^{\alpha}}\frac{\partial X^{\mu}}{\partial X'^{\beta}}{}^{\rho}N'^{\sigma\alpha\beta}(X')\right) \end{aligned} \tag{19} $$

Since it has been supposed that ${}^{\rho}N'^{\sigma\alpha\beta}(X')$ can be neglected, (19) reduces to

$$ N^{\lambda\nu\mu}(X)=\frac{\partial X^{\lambda}}{\partial X'^{\sigma}}\frac{\partial X^{\nu}}{\partial X'^{\alpha}}\frac{\partial X^{\mu}}{\partial X'^{\beta}}N'^{\sigma\alpha\beta}(X') \tag{19'} $$

The transformation formula of $S^{\lambda\nu}$ can be derived from (8), (19′), and (14); we get

$$ \begin{aligned} S^{\lambda\nu}(X)&=\frac{N^{\lambda\nu 0}(X)}{U^{0}}\\ &=\frac{1}{U^{0}}\frac{\partial X^{\lambda}}{\partial X'^{\sigma}}\frac{\partial X^{\nu}}{\partial X'^{\alpha}}\frac{\partial X^{0}}{\partial X'^{\beta}}N'^{\sigma\alpha\beta}(X')\\ &=\frac{1}{U^{0}}\frac{\partial X^{\lambda}}{\partial X'^{\sigma}}\frac{\partial X^{\nu}}{\partial X'^{\alpha}}\frac{\partial X^{0}}{\partial X'^{\beta}}[U'^{\beta}S'^{\sigma\alpha}(X')+2{}^{\beta}M'^{[\sigma\alpha]}(X')]\\ &=\frac{\partial X^{\lambda}}{\partial X'^{\sigma}}\frac{\partial X^{\nu}}{\partial X'^{\alpha}}[S'^{\sigma\alpha}(X')+\frac{2}{U^{0}}\frac{\partial X^{0}}{\partial X'^{\beta}}{}^{\beta}M'^{[\sigma\alpha]}(X')] \end{aligned} \tag{20} $$

In order to derive the transformation formula of $L^{\lambda\nu}$, I write first a relation which enables ${}^{\lambda}M^{\nu\mu}(X)$ to be connected with $U^{\nu}(X)$, $L^{\lambda\nu}(X)$, etc.:

$$ \begin{aligned} {}^{\lambda}M^{\nu\mu}=&\frac{1}{2}\frac{U^{\lambda}}{U^{0}}(U^{\nu}L^{\mu 0}+U^{\mu}L^{\nu 0})+\tfrac{1}{2}(U^{\nu}L^{\lambda\mu}+U^{\mu}L^{\lambda\nu})\\ &+{}^{\lambda}M^{[\nu\mu]}-{}^{\nu}M^{[\mu\lambda]}-{}^{\mu}M^{[\nu\lambda]}+U^{\lambda}({}^{\nu}M^{[\mu 0]}+{}^{\mu}M^{[\nu 0]}) \end{aligned} \tag{21} $$

This relation is proved in Appendix B. The transformation formula of $L^{\lambda\nu}$ can be derived from (7), (18), and (21); we get

$$ \begin{aligned} L^{\lambda\nu}(X)&=\frac{{}^{\lambda}M^{\nu 0}(X)-{}^{\nu}M^{\lambda 0}(X)}{U^{0}}\\ &=\frac{1}{U^{0}}\left[\frac{\partial X^{\nu}}{\partial X'^{\alpha}}\frac{\partial X^{0}}{\partial X'^{\beta}}\left(\frac{\partial X^{\lambda}}{\partial X'^{\rho}}-\frac{U^{\lambda}}{U^{0}}\frac{\partial X^{0}}{\partial X'^{\rho}}\right)-\frac{\partial X^{\lambda}}{\partial X'^{\alpha}}\frac{\partial X^{0}}{\partial X'^{\beta}}\left(\frac{\partial X^{\nu}}{\partial X'^{\rho}}-\frac{U^{\lambda}}{U^{0}}\frac{\partial X^{0}}{\partial X'^{\rho}}\right)\right] \end{aligned} $$

$$\times\left\{\frac{1}{2}\frac{U'^{\rho}}{U'^{0}}[U'^{\alpha}L'^{\beta 0}(X')+U'^{\beta}L'^{\alpha 0}(X')]+\tfrac{1}{2}[U'^{\alpha}L'^{\rho\beta}(X')+U'^{\beta}L'^{\rho\alpha}(X')]\right.$$

$$\left.+{}^{\rho}M'^{[\alpha\beta]}-{}^{\alpha}M'^{[\beta\rho]}-{}^{\beta}M'^{[\alpha\rho]}+U'^{\rho}({}^{\alpha}M'^{[\beta 0]}+{}^{\beta}M'^{[\alpha 0]})\right\}$$

$$=\frac{\partial X^{\lambda}}{\partial X'^{\rho}}\frac{\partial X^{\nu}}{\partial X'^{\alpha}}L'^{\rho\alpha}(X')+\frac{1}{U^{0}}\left[\frac{\partial X^{\nu}}{\partial X'^{\alpha}}\frac{\partial X^{0}}{\partial X'^{\beta}}\left(\frac{\partial X^{\lambda}}{\partial X'^{\rho}}-\frac{U^{\lambda}}{U^{0}}\frac{\partial X^{0}}{\partial X'^{\rho}}\right)\right.$$

$$\left.-\frac{\partial X^{\lambda}}{\partial X'^{\alpha}}\frac{\partial X^{0}}{\partial X'^{\beta}}\left(\frac{\partial X^{\nu}}{\partial X'^{\rho}}-\frac{U^{\lambda}}{U^{0}}\frac{\partial X^{0}}{\partial X'^{\rho}}\right)\right][{}^{\rho}M'^{[\alpha\beta]}-{}^{\alpha}M'^{[\beta\rho]}-{}^{\beta}M'^{[\alpha\rho]}$$

$$+U'^{\rho}({}^{\alpha}M'^{[\beta 0]}+{}^{\beta}M'^{[\alpha 0]})] \tag{22}$$

It is evident from (20) and (22) that $S^{\lambda\nu}$ and $L^{\lambda\nu}$ all are not tensors if the energy-momentum tensor is asymmetric. But in either case the energy-momentum tensor is symmetric or its antisymmetric part can be neglected on account of ${}^{\rho}M'^{[\alpha\beta]}=0$, (20) and (21) become

$$S^{\lambda\nu}(X)=\frac{\partial X^{\lambda}}{\partial X'^{\sigma}}\frac{\partial X^{\nu}}{\partial X'^{\alpha}}S'^{\sigma\alpha}(X') \tag{23}$$

and

$$L^{\lambda\nu}(X)=\frac{\partial X^{\lambda}}{\partial X'^{\rho}}\frac{\partial X^{\nu}}{\partial X'^{\alpha}}L'^{\rho\alpha}(X') \tag{24}$$

respectively, i.e., they are tensors.

Since physical theory requires always that angular momenta $S^{\lambda\nu}$ and $L^{\lambda\nu}$ must be tensors, then, in the case of asymmetric energy-momentum the definitions of $S^{\lambda\nu}$ and $L^{\lambda\nu}$ for test bodies must be revised. If the definitions (7) and (8) are preserved, the energy-momentum tensor must be symmetric or its antisymmetric part must be neglected; I shall consider only the case of $\mathfrak{T}^{[\nu\mu]}=0$ in the following paragraph.

In I the momentum of test particles was defined by

$$p_{\nu}(X)=\int \mathfrak{T}_{\nu}{}^{0}(x)\,d^{3}x \tag{25}$$

Let

$$p^{\nu}(X)=g^{\nu\mu}(X)p_{\mu}(X)$$

Since ${}^{\lambda}M^{\nu\mu}=0$ for test particles, it is easy to demonstrate

$$p^{\nu}(X)=\int g^{\nu\mu}(x)\mathfrak{T}_{\mu}{}^{0}(x)\,d^3x=\int \mathfrak{T}^{\nu 0}(x)\,d^3x \tag{26}$$

So the momentum of test particles can also be defined by (26).

Utilizing (4), we can write the definition (26) as

$$p^{\nu}(X)=\frac{M^{\nu 0}(X)}{U^0} \tag{27}$$

The transformation formula of $M^{\nu 0}(X)$ can be obtained from (17) and (11):

$$\begin{aligned}
M^{\nu 0}(X)=&\frac{\partial X^{\nu}}{\partial X'^{\alpha}}\frac{\partial X^{0}}{\partial X'^{\beta}}M'^{\alpha\beta}(X')+\frac{\partial}{\partial X'^{\lambda}}\left(\frac{\partial X^{\nu}}{\partial X'^{\alpha}}\frac{\partial X^{0}}{\partial X'^{\beta}}\right){}^{\lambda}M'^{\alpha\beta}(X')\\
&-\frac{d}{dS'}\left[\frac{1}{U^0}\frac{\partial X^0}{\partial X'^{\lambda}}\frac{\partial X^{\nu}}{\partial X'^{\alpha}}\frac{\partial X^{0}}{\partial X'^{\beta}}{}^{\lambda}M'^{\alpha\beta}(X')\right]\\
=&\frac{\partial X^{\nu}}{\partial X'^{\alpha}}\frac{\partial X^{0}}{\partial X'^{\beta}}\left[U'^{\beta}\frac{M'^{\alpha 0}(X')}{U'^0}+\frac{d}{dS'}\left(\frac{{}^{\beta}M'^{\alpha 0}(X')}{U'^0}\right)\right.\\
&\left.+\left(\left\{\begin{matrix}\alpha\\ \sigma\rho\end{matrix}\right\}'+A'^{\alpha}{}_{\sigma\rho}\right){}^{\beta}M'^{\sigma\beta}(X')\right]+\frac{\partial}{\partial X'^{\lambda}}\left(\frac{\partial X^{\nu}}{\partial X'^{\alpha}}\frac{\partial X^{0}}{\partial X'^{\beta}}\right){}^{\lambda}M'^{\alpha\beta}(X')\\
&-\frac{d}{dS'}\left[\frac{1}{U^0}\frac{\partial X'^0}{\partial X^{\lambda}}\frac{\partial X^{\nu}}{\partial X'^{\alpha}}\frac{\partial X^{0}}{\partial X'^{\beta}}{}^{\lambda}M'^{\alpha\beta}(X')\right]
\end{aligned} \tag{28}$$

Owing to ${}^{\lambda}M'^{\alpha\beta}=0$ for the test particles, (28) becomes

$$M^{\nu 0}(X)=\frac{U^0}{U'^0}\frac{\partial X^{\nu}}{\partial X'^{\alpha}}M'^{\alpha 0}(X') \tag{29}$$

Hence

$$p^{\nu}(X)=\frac{M^{\nu 0}(X)}{U^0}=\frac{\partial X^{\nu}}{\partial X'^{\alpha}}\frac{M'^{\alpha 0}(X')}{U'^0}=\frac{\partial X^{\nu}}{\partial X'^{\alpha}}p'^{\alpha}(X') \tag{30}$$

For the test particles, this means that $p^{\nu}(X)$ is a 4-vector. We point out that the relation (30) does not exist for test bodies because their ${}^{\lambda}M'^{\alpha\beta}\neq 0$ in (28).

Since physical theory requires always that the momentum $p^{\nu}$ must be a 4-vector, the definition of momentum for test bodies ought to be revised. Papapetrou (1951) demonstrated the following relation for test bodies with symmetric energy-momentum tensor:

$$\frac{1}{U^0}\left[M^{\nu 0}(x)+\begin{Bmatrix}\nu\\ \rho\sigma\end{Bmatrix}U^\rho L^{\sigma 0}(X)\right]$$
$$=\frac{\partial X^\nu}{\partial X'^\alpha}\frac{1}{U'^0}\left[M'^{\alpha 0}(X')+\begin{Bmatrix}\alpha\\ \delta\eta\end{Bmatrix}' U'^\delta L'^{\eta 0}(X')\right] \tag{31}$$

Using this relation, I make the following definition of the momentum 4-vector for test bodies when $\mathfrak{T}^{[\nu\mu]}=0$:

$$p^\nu=\frac{1}{U^0}\left(M^{\nu 0}+\begin{Bmatrix}\nu\\ \rho\sigma\end{Bmatrix}U^\rho L^{\sigma 0}\right) \tag{32}$$

It is evident that the quantity $m=p^\nu U_\mu$ is a scalar. Owing to $L^{\nu\mu}=-L^{\mu\nu}$, we have

$$m=p^\nu U_\nu=p^\nu U_\nu+U_\nu U_\mu\frac{DL^{\nu\mu}}{DS}$$
$$=\left(p^\nu+U_\mu\frac{DL^{\nu\mu}}{DS}\right)U_\nu=P^\nu U_\nu \tag{33}$$

where

$$P^\nu=p^\nu+U_\mu\frac{DL^{\nu\mu}}{DS}=\frac{1}{U^0}\left(M^{\nu 0}+\begin{Bmatrix}\nu\\ \rho\sigma\end{Bmatrix}U^\rho L^{\sigma 0}\right)+U^\mu\frac{DL^{\nu\mu}}{DS}$$

and

$$\frac{DL^{\nu\mu}}{DS}:=\frac{dL^{\nu\mu}}{ds}+\begin{Bmatrix}\nu\\ \eta\tau\end{Bmatrix}L^{\eta\mu}U^\tau+\begin{Bmatrix}\mu\\ \eta\tau\end{Bmatrix}L^{\nu\eta}U^\tau \tag{34}$$

Papapetrou (1951) introduced the quantity $P^\nu$ in his equations of motion for test bodies and so did Carmeli (1982) and others. I would not like to introduce $P^\nu$ and shall use only $p^\nu$ because $p^\nu$ is simpler than $P^\nu$.

## 4. THE MOMENTUM AND ANGULAR MOMENTUM EQUATIONS OF MOTION FOR TEST BODIES WHEN $\mathfrak{T}^{[\nu\mu]}=0$

In case the energy-momentum tensor is symmetric or its antisymmetric part can be neglected, $\mathfrak{T}^{[\nu\mu]}=0$; hence

$$M^{\nu\mu}=M^{\mu\nu};\qquad {}^\lambda M^{[\alpha\beta]}=0;\qquad A^\nu{}_{\sigma\mu}M^{\sigma\mu}=0$$

Then equations (10), (11), (13), (14), and (21) become

$$\frac{d}{dS}\left(\frac{M^{\nu 0}}{U^0}\right)+\begin{Bmatrix}\nu\\ \sigma\mu\end{Bmatrix}M^{\sigma\mu}+\left(\frac{\partial}{\partial X^\lambda}\begin{Bmatrix}\nu\\ \sigma\mu\end{Bmatrix}\right){}^{\lambda}M^{\sigma\mu}=\tfrac{1}{2}R_{\alpha\beta\mu}{}^{\nu}U^{\mu}S^{\alpha\beta} \tag{35}$$

$$M^{\nu\mu}=U^{\mu}\frac{M^{\nu 0}}{U^0}+\frac{d}{dS}\left(\frac{{}^{\mu}M^{\nu 0}}{U^0}\right)+\begin{Bmatrix}\nu\\ \sigma\tau\end{Bmatrix}{}^{\mu}M^{\sigma\tau} \tag{36}$$

$$\frac{d}{dS}S^{\lambda\nu}=-\Gamma^{\lambda}_{\sigma\mu}N^{\sigma\nu\mu}-\Gamma^{\nu}{}_{\sigma\mu}N^{\lambda\sigma\mu} \tag{37}$$

$$N^{\lambda\nu\mu}=U^{\mu}S^{\lambda\nu} \tag{38}$$

and

$${}^{\lambda}M^{\nu\mu}=\frac{1}{2}\frac{U^{\lambda}}{U^0}(U^{\nu}L^{\mu 0}+U^{\mu}L^{\nu 0})+\tfrac{1}{2}(U^{\nu}L^{\lambda\mu}+U^{\mu}L^{\lambda\nu})$$

respectively.

From (36), (7), and $M^{\nu\mu}=M^{\mu\nu}$ we get an important relation:

$$\frac{U^{\mu}}{U^0}M^{\nu 0}-\frac{U^{\nu}}{U^0}M^{\mu 0}+\frac{d}{dS}(L^{\mu\nu})+\begin{Bmatrix}\nu\\ \sigma\tau\end{Bmatrix}{}^{\mu}M^{\sigma\tau}-\begin{Bmatrix}\mu\\ \sigma\tau\end{Bmatrix}{}^{\nu}M^{\sigma\tau}=0 \tag{40}$$

We can also write (36) in the form

$$M^{\mu\nu}=U^{\mu}\frac{M^{\nu 0}}{U^0}+\frac{d}{dS}\left(\frac{{}^{\mu}M^{\nu 0}}{U^0}\right)+\begin{Bmatrix}\mu\\ \sigma\tau\end{Bmatrix}{}^{\mu}M^{\sigma\tau} \tag{36'}$$

Therefore

$$M^{\mu 0}=U^{\mu}\frac{M^{00}}{U^0}+\frac{d}{dS}\left(\frac{{}^{\mu}M^{00}}{U^0}\right)+\begin{Bmatrix}0\\ \sigma\tau\end{Bmatrix}{}^{\mu}M^{\sigma\tau} \tag{41}$$

With the help of (41), (7), and ${}^{0}M^{\sigma\tau}=0$, equation (40) can be transformed into

$$\frac{d}{dS}L^{\mu\nu}+\frac{U^{\mu}}{U^0}\frac{dL^{\nu 0}}{dS}-\frac{U^{\nu}}{U^0}\frac{dL^{\mu 0}}{dS}+\left(\begin{Bmatrix}\nu\\ \sigma\tau\end{Bmatrix}-\frac{U^{\nu}}{U^0}\begin{Bmatrix}0\\ \sigma\tau\end{Bmatrix}\right){}^{\mu}M^{\sigma\tau}$$
$$+-\left(\begin{Bmatrix}\mu\\ \sigma\tau\end{Bmatrix}-\frac{U^{\mu}}{U^0}\begin{Bmatrix}0\\ \sigma\tau\end{Bmatrix}\right){}^{\nu}M^{\sigma\tau}=0 \tag{42}$$

It is now possible to derive the equation of motion in a covariant form. From (37) and (38) we get

$$\frac{d}{dS}S^{\lambda\nu}=-(\Gamma^{\lambda}{}_{\sigma\mu}S^{\sigma\nu}+\Gamma^{\nu}{}_{\sigma\mu}S^{\lambda\sigma})U^{\mu} \tag{43}$$

This is the generalized spin angular momentum equation of motion for test bodies.

The rotation angular momentum equation of motion for test bodies can be derived from (42), (34) and (39) without difficulty:

$$\frac{DL^{\mu\nu}}{DS}+\frac{U^{\mu}}{U^{0}}\frac{DL^{\nu 0}}{DS}-\frac{U^{\nu}}{U^{0}}\frac{DL^{\mu 0}}{DS}=0 \tag{44}$$

Since we have the relation (Papapetrou, 1951)

$$p^{\mu}=\frac{1}{U^{0}}\left(M^{\mu 0}+\begin{Bmatrix}\mu\\ \sigma\lambda\end{Bmatrix}L^{\sigma 0}U^{\lambda}\right)=\frac{U^{\mu}}{(U^{0})^{2}}\left(M^{00}+\begin{Bmatrix}0\\ \sigma\lambda\end{Bmatrix}L^{\sigma 0}U^{\lambda}\right)+\frac{1}{U^{0}}\frac{DL^{\mu 0}}{DS} \tag{45}$$

we can also write equation (44) in the form

$$\frac{DL^{\mu\nu}}{DS}+U^{\mu}p^{\nu}-U^{\nu}p^{\mu}=0 \tag{44'}$$

or

$$\frac{dL^{\mu\nu}}{dS}=-\left(\begin{Bmatrix}\mu\\ \eta\tau\end{Bmatrix}L^{\eta\nu}+\begin{Bmatrix}\nu\\ \eta\tau\end{Bmatrix}L^{\mu\eta}\right)U^{\tau}+U^{\nu}p^{\mu}-U^{\mu}p^{\nu} \tag{44''}$$

From (35), (32), (36), and (39) we can obtain the momentum equation of motion for test bodies:

$$\frac{dp^{\nu}}{dS}=-\begin{Bmatrix}\nu\\ \sigma\mu\end{Bmatrix}p^{\sigma}U^{\mu}+\tfrac{1}{2}R_{\alpha\beta\mu}{}^{\nu}S^{\alpha\beta}U^{\mu}+\tfrac{1}{2}R_{\alpha\beta\mu}{}^{\nu}(\{\cdot\})L^{\alpha\beta}U^{\mu} \tag{46}$$

after some elementary calculations, where

$$R_{\alpha\beta\mu}{}^{\nu}=g_{\alpha\lambda}g^{\nu\tau}R^{\lambda}{}_{\beta\mu\tau}$$

$$R^{\lambda}{}_{\beta\mu\tau}=\Gamma^{\lambda}{}_{\beta\tau,\mu}-\Gamma^{\lambda}{}_{\beta\mu,\tau}+\Gamma^{\lambda}{}_{\rho\mu}\Gamma^{\rho}{}_{\beta\tau}-\Gamma^{\lambda}{}_{\rho\tau}\Gamma^{\rho}{}_{\beta\mu}$$

$$R_{\alpha\beta\mu}{}^{\nu}(\{\cdot\})=g_{\alpha\lambda}g^{\nu\tau}R^{\lambda}{}_{\beta\mu\tau}(\{\cdot\})$$

$$R^{\lambda}{}_{\beta\mu\tau}(\{\cdot\})=\frac{\partial}{\partial X^{\mu}}\begin{Bmatrix}\lambda\\ \beta\tau\end{Bmatrix}-\frac{\partial}{\partial X^{\tau}}\begin{Bmatrix}\lambda\\ \beta\mu\end{Bmatrix}+\begin{Bmatrix}\lambda\\ \rho\mu\end{Bmatrix}\begin{Bmatrix}\rho\\ \beta\tau\end{Bmatrix}-\begin{Bmatrix}\lambda\\ \rho\tau\end{Bmatrix}\begin{Bmatrix}\rho\\ \beta\mu\end{Bmatrix}$$

## APPENDIX A

Since $U^0 = dX^0/dS = dx^0/dS$ and $dS = dS'$, under the infinitesimal coordinate transformation $x^\mu \to x'^\mu = x^\mu + \xi^\mu(x)$ we have

$$
\begin{aligned}
Q^{\mu_1\mu_2\cdots}_{\alpha_1\alpha_2\cdots}(X) &= U^0 \int \mathscr{D}^{\mu_1\mu_2\cdots}_{\alpha_1\alpha_2\cdots}\, d^3x \\
&= \frac{1}{dS'} \int dx'^0 \frac{\partial x^{\mu_1}}{\partial x'^{\nu_1}} \frac{\partial x^{\mu_2}}{\partial x'^{\nu_2}} \cdots \frac{\partial x'^{\beta_1}}{\partial x^{\alpha_1}} \\
&\quad \times \frac{\partial x'^{\beta_2}}{\partial x^{\alpha_2}} \cdots \mathscr{D}'^{\nu_1\nu_2\cdots}_{\beta_1\beta_2\cdots}(x')\, d^3x'
\end{aligned}
\tag{A1}
$$

and

$$
\begin{aligned}
{}^\lambda Q^{\mu_1\mu_2\cdots}_{\alpha_1\alpha_2\cdots}(X) &= U^0 \int \delta x^\lambda \mathscr{D}^{\mu_1\mu_2\cdots}_{\alpha_1\alpha_2\cdots}(x)\, d^3x \\
&= \frac{1}{dS'} \int dx'^0\, \delta x'^\rho \frac{\partial x^\lambda}{\partial x'^\rho} \frac{\partial x^{\mu_1}}{\partial x'^{\nu_1}} \\
&\quad \times \frac{\partial x^{\mu_2}}{\partial x'^{\nu_2}} \cdots \frac{\partial x'^{\beta_1}}{\partial x^{\alpha_1}} \frac{\partial x'^{\beta_2}}{\partial x^{\alpha_2}} \cdots \mathscr{D}'^{\nu_1\nu_2\cdots}_{\beta_1\beta_2\cdots}(x')\, d^3x'
\end{aligned}
\tag{A2}
$$

where $\delta x^\lambda = x^\lambda - X^\lambda$, $\delta x'^\rho = x'^\rho - X'^\rho$.

When $x^\mu \to x'^\mu$, the world tube $L$ and the representative point $X$ of a test body are transformed into $L'$ and $X'$, respectively, (see Figure 1) and the hyperplane $AB$ is transformed into the hypersurface $A'B'$. Note that $x'^0 \neq X'^0$ on $A'B'$, although $x^0 = X^0$ on $AB$, but we can draw a hyperplane $CD$ through $X'$; and enable that the time coordinate $y^0$ of every point $y$ on it satisfies $y^0 = X'^0$.

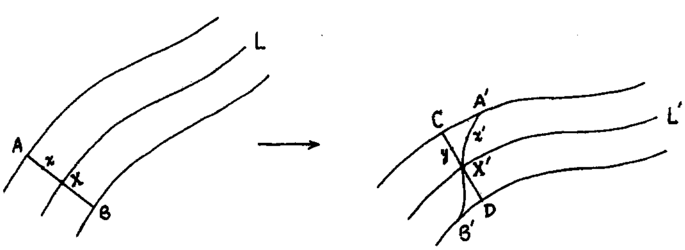


Fig. 1

Now consider two points $y$ and $x'$ on one world line; it is easy to find

$$x'^{\mu} = y^{\mu} + u'^{\mu} \frac{\delta x'^{0}}{u'^{0}} \tag{A3}$$

where

$$u'^{\mu} = \frac{dx'^{\mu}}{ds'}, \qquad ds' = g_{\mu\nu}\, dx^{\mu}\, dx^{\nu}$$

For macroscopic bodies there exist always the relations $u^{i} \ll u^{0}$ $(i = 1, 2, 3)$; using these relations, we can change (A3) to

$$x'^{\mu} \simeq y^{\mu} - \frac{u'^{\mu}}{u^{0}} \frac{\partial x^{0}}{\partial x'^{\sigma}} \delta y^{\sigma} \tag{A4}$$

where $\delta y^{\sigma} = y^{\sigma} - Y^{\sigma}$ and $Y^{\sigma} = X^{\sigma}$.

From (A1), (A2), and (A4), one can prove the transformation formulas (15) and (16) after carrying out the integration over the hyperplane $CD$.

## APPENDIX B

From the relation (12) we have

$$U^{0}({}^{\lambda}M^{\nu\mu} + {}^{\mu}M^{\nu\lambda}) = U^{\lambda}({}^{\mu}M^{\nu 0}) + U^{\mu}({}^{\lambda}M^{\nu 0})$$

Two other relations may be written by taking the cyclic permutations of the indices $\lambda\nu\mu$. Adding the first and second of these relations and subtracting the third, we obtain

$$\begin{aligned} 2U^{0}({}^{\lambda}M^{\nu\mu} - {}^{\lambda}M^{[\nu\mu]} + {}^{\nu}M^{[\mu\lambda]} + {}^{\mu}M^{[\nu\lambda]}) \\ = U^{\lambda}({}^{\nu}M^{(\mu 0)} + {}^{\mu}M^{(\nu 0)}) + U^{\lambda}({}^{\nu}M^{[\mu 0]} + {}^{\mu}M^{[\nu 0]}) \\ + U^{\nu}({}^{\lambda}M^{[\mu 0]} - {}^{\mu}M^{[\lambda 0]}) + U^{\mu}({}^{\lambda}M^{[\nu 0]} - {}^{\nu}M^{[\lambda 0]}) \end{aligned} \tag{B1}$$

From (B1), (7), and the relation

$$\begin{aligned} U^{0}({}^{\nu}M^{(\mu 0)} + {}^{\mu}M^{(\nu 0)} &= U^{\nu}M^{\mu 00} + U^{\mu}M^{\nu 00} \\ &= U^{0}(U^{\nu}L^{\mu 0} + U^{\mu}L^{\nu 0}) \end{aligned}$$

one can easily prove the transformation formula (21).

## ADDITIONAL REMARKS

From (43) and (44″) we can get the total angular momentum equation of motion for test bodies:

$$\frac{d}{dS}(L^{\alpha\beta}+S^{\alpha\beta})=-\left(\begin{Bmatrix}\alpha\\ \sigma\mu\end{Bmatrix}L^{\sigma\beta}+\begin{Bmatrix}\beta\\ \sigma\mu\end{Bmatrix}L^{\alpha\sigma}+\Gamma^{\alpha}{}_{\sigma\mu}S^{\sigma\beta}+\Gamma^{\beta}{}_{\sigma\mu}S^{\alpha\sigma}\right)U^{\mu}$$
$$+p^{\alpha}U^{\beta}-p^{\beta}U^{\alpha} \qquad \text{(R1)}$$

This equation also can be derived in the following way. When the energy-momentum tensor is symmetric, equations (1′) and (2′) reduce to

$$\mathfrak{T}^{\nu\mu}{}_{,\mu}+\begin{Bmatrix}\nu\\ \sigma\mu\end{Bmatrix}\mathfrak{T}^{\sigma\mu}-\tfrac{1}{2}\mathfrak{C}^{\alpha\beta\mu}R_{\alpha\beta\mu}{}^{\nu}=0 \qquad \text{(R2)}$$

and

$$\mathfrak{C}^{\alpha\beta\mu}{}_{,\mu}=-\Gamma^{\alpha}{}_{\sigma\mu}\mathfrak{C}^{\sigma\beta\mu}-\Gamma^{\beta}{}_{\sigma\mu}\mathfrak{C}^{\alpha\sigma\mu} \qquad \text{(R3)}$$

Since

$$\frac{\partial}{\partial x^{\mu}}(x^{\alpha}\mathfrak{T}^{\mu\beta}-x^{\beta}\mathfrak{T}^{\mu\alpha})=\mathfrak{T}^{\alpha\beta}-\mathfrak{T}^{\beta\alpha}+x^{\alpha}\mathfrak{T}^{\mu\beta}{}_{,\mu}-x^{\beta}\mathfrak{T}^{\mu\alpha}{}_{,\mu}$$

we have

$$\frac{\partial}{\partial x^{\mu}}(x^{\alpha}\mathfrak{T}^{\beta\mu}-x^{\beta}\mathfrak{T}^{\alpha\mu})=x^{\alpha}\mathfrak{T}^{\beta\mu}{}_{,\mu}-x^{\beta}\mathfrak{T}^{\alpha\mu}{}_{,\mu} \qquad \text{(R4)}$$

Utilizing (R4) and (R2), we can write (R3) in the form

$$\frac{\partial}{\partial x^{\mu}}(\mathfrak{C}^{\alpha\beta\mu}+x^{\alpha}\mathfrak{T}^{\beta\mu}-x^{\beta}\mathfrak{T}^{\alpha\mu})$$
$$=-\Gamma^{\alpha}{}_{\sigma\mu}\mathfrak{C}^{\sigma\beta\mu}-\Gamma^{\beta}{}_{\sigma\mu}\mathfrak{C}^{\alpha\sigma\mu}-x^{\alpha}\left(\begin{Bmatrix}\beta\\ \sigma\mu\end{Bmatrix}\mathfrak{T}^{\sigma\mu}-\tfrac{1}{2}\mathfrak{C}^{\lambda\rho\mu}R_{\lambda\rho\mu}{}^{\beta}\right)$$
$$+x^{\beta}\left(\begin{Bmatrix}\alpha\\ \sigma\mu\end{Bmatrix}\mathfrak{T}^{\sigma\mu}-\tfrac{1}{2}\mathfrak{C}^{\lambda\rho\mu}R_{\lambda\rho\mu}{}^{\alpha}\right) \qquad \text{(R5)}$$

It is not difficult to derive the following relation from (R5):

$$\frac{d}{dS}(S^{\alpha\beta}+L^{\alpha\beta})+\frac{U^\alpha}{U^0}M^{\beta 0}-\frac{U^\beta}{U^0}M^{\alpha 0}=-(\Gamma^\alpha{}_{\sigma\mu}S^{\sigma\beta}+\Gamma^\beta{}_{\sigma\mu}S^{\alpha\sigma})U^\mu$$
$$+\begin{Bmatrix}\alpha\\ \sigma\mu\end{Bmatrix}{}^\beta M^{\sigma\mu}-\begin{Bmatrix}\beta\\ \sigma\mu\end{Bmatrix}{}^\alpha M^{\sigma\mu} \quad \text{(R6)}$$

This equation is the sum of (40) and (43), which is equivalent to (R1).

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